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On Legendre functions of imaginary degree and associated integral transforms

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Abstract

New integral representations, asymptotic formulas, and series expansions in powers of $\tanh(t/2)$ are obtained for the imaginary and real parts of the Legendre function $P_{i\xi}(\cosh t)$. Coefficients of these series expansions are orthogonal polynomials in the real variable ξ. A number of relations for these orthogonal polynomials are obtained on the basis of the generating function. Several inversion theorems are proven for the integral transforms involving the Legendre function of imaginary degree. In many cases it is preferable to employ these transforms, than Mehler–Fok transforms, since conditions placed on functions are less restrictive.

Keywords: Legendre function; Pollaczek polynomials; Integral transforms

1. Introduction

The Legendre function $P_{i\xi}(\cosh t)$ and integral transforms with kernels involving this function often occur in various problems of applied mathematics. Legendre functions are investigated thoroughly (see [1, Vol. 1] and [2–4]). The subject of this paper is study of interesting particulars as the degree of the Legendre function is imaginary.

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New integral representations for $P_{i\xi}(\cosh t)$ are found in Section 2. Series expansions in powers of $\tanh(t/2)$ are derived from them. In contrast to the known expansions [1,4], coefficients of these series are orthogonal polynomials in the parameter ξ. These polynomials are the special cases of the Pollaczek polynomials [1, Vol. 2]. A number of relations for the coefficients are established on the basis of a generating function. In the conclusion of the section, some important integrals involving $P_{i\xi}(\cosh t)$ are evaluated by these expansions.

Asymptotic formulas, as $\xi \to \infty$, are treated in Section 3. The uniform asymptotic series obtained involve Bessel functions of the first kind. We note that a similar construction for Legendre polynomials is well known.

Integral transforms with kernels containing the Legendre function of imaginary degree are dealt with in Section 4. These transforms are related to the generalized Mehler–Fok transforms which were discussed by Vilenkin [4,5]. Here we prove several inversion theorems for functions belonging to classes L_1 and L_2 which often arise in applications.

2. Legendre functions of imaginary degree

Consider the Legendre function of imaginary degree

$$P_{i\xi}(\cosh t) = P^+(\xi, t) + i P^-(\xi, t),$$

where $\xi \geqslant 0$ and $P^\pm(\xi, t)$ are real functions for real t. It follows from the relations [1, Vol. 1]

$$\sinh t\, P'_\nu(\cosh t) = \nu\big[\cosh t\, P_\nu(\cosh t) - P_{\nu-1}(\cosh t)\big],$$
$$P_{-\nu-1}(\cosh t) = P_\nu(\cosh t),$$

that

$$\frac{dP^-(\xi,t)}{dt} = \xi \tanh\frac{t}{2} P^+(\xi,t), \qquad \frac{dP^+(\xi,t)}{dt} = -\xi \coth\frac{t}{2} P^-(\xi,t). \tag{2.1}$$

Hence the functions $P^\pm(\xi, t)$ obey the hypergeometric differential equations

$$\frac{d^2P^\pm(\xi,t)}{dt^2} \pm \frac{1}{\sinh t}\frac{dP^\pm(\xi,t)}{dt} + \xi^2 P^\pm(\xi,t) = 0 \tag{2.2}$$

subject to the initial conditions

$$P^+(\xi,0) = 1, \quad \frac{dP^+(\xi,0)}{dt} = 0, \qquad P^-(\xi,0) = 0, \quad \frac{dP^-(\xi,0)}{d(\cosh t)} = \frac{\xi}{2}.$$

Integral representations of $P^\pm(\xi, t)$ are found by equating the real and imaginary parts of the integral representation of $P_{i\xi}(\cosh t)$ [1, Vol. 1]:

$$P^+(\xi,t) = \frac{\sqrt{2}}{\pi}\int_0^t \frac{\cosh(x/2)\cos\xi x\, dx}{\sqrt{\cosh t - \cosh x}},$$

$$P^-(\xi,t) = \frac{\sqrt{2}}{\pi}\int_0^t \frac{\sinh(x/2)\sin\xi x\, dx}{\sqrt{\cosh t - \cosh x}}. \tag{2.3}$$

New integral representations can be derived from these by examining the sum of the contour integrals

$$\sum_{n=1}^{2}\oint_{L_n}\left[\frac{\cosh(z/2)}{\sqrt{\cosh t-\cosh z}}+\frac{(-1)^n}{\sqrt{2}}\right]\exp(\pm i\xi z)\,dz=0,\qquad t>0,$$

where the path L_n consists of the arc $|z-(-1)^n t|=\varepsilon$, $0\leqslant \arg z\leqslant \pi$ and segments of the rectangle with vertices $(0,0)$, $(0,i\pi)$, $((-1)^n R,i\pi)$, $((-1)^n R,0)$. As $R\to\infty$, we obtain

$$\pm\frac{\pi}{\sqrt{2}}P^+(\xi,t)=\int_t^\infty\left[\frac{\cosh(x/2)}{\sqrt{\cosh x-\cosh t}}-\frac{1}{\sqrt{2}}\right]\sin\xi x\,dx+\frac{\cos\xi t-\exp(\mp\pi\xi)}{\xi\sqrt{2}}$$
$$-\exp(\mp\pi\xi)\int_0^\infty\left[\frac{\sinh(x/2)}{\sqrt{\cosh t+\cosh x}}-\frac{1}{\sqrt{2}}\right]\sin\xi x\,dx.$$

It follows from this relation that

$$\pi\coth\pi\xi\,P^+(\xi,t)=\int_t^\infty\left[\frac{\sqrt{2}\cosh(x/2)}{\sqrt{\cosh x-\cosh t}}-1\right]\sin\xi x\,dx+\frac{\cos\xi t}{\xi},\tag{2.4}$$

$$\frac{\pi}{\sinh\pi\xi}P^+(\xi,t)=\int_0^\infty\left[\frac{\sqrt{2}\sinh(x/2)}{\sqrt{\cosh t+\cosh x}}-1\right]\sin\xi x\,dx+\frac{1}{\xi}$$
$$=\frac{\sqrt{2}\cosh^2(t/2)}{\xi}\int_0^\infty\frac{\cosh(x/2)\cos\xi x\,dx}{(\cosh t+\cosh x)^{3/2}}.\tag{2.5}$$

In the same way one can find

$$\pi\coth\pi\xi\,P^-(\xi,t)=\int_t^\infty\left[1-\frac{\sqrt{2}\sinh(x/2)}{\sqrt{\cosh x-\cosh t}}\right]\cos\xi x\,dx-\frac{\sin\xi t}{\xi},\tag{2.6}$$

$$\frac{\pi}{\sinh\pi\xi}P^-(\xi,t)=\int_0^\infty\left[1-\frac{\sqrt{2}\cosh(x/2)}{\sqrt{\cosh t+\cosh x}}\right]\cos\xi x\,dx$$
$$=\frac{\sqrt{2}\sinh^2(t/2)}{\xi}\int_0^\infty\frac{\sinh(x/2)\sin\xi x\,dx}{(\cosh t+\cosh x)^{3/2}}.\tag{2.7}$$

These integral representations enable us to determine series expansions for $P^{\pm}(\xi,t)$. Inserting the binomial series

$$\frac{1}{(\cosh t+\cosh x)^{3/2}}=\frac{1}{2^{3/2}\cosh^3(t/2)\cosh^3(x/2)}\sum_{n=0}^\infty\frac{(2n+1)!!}{(2n)!!}\tanh^{2n}\frac{t}{2}\tanh^{2n}\frac{x}{2}$$

into (2.5), (2.7) yields for $P^+(\xi,t)$ and $P^-(\xi,t)$ expansions in ascending powers of $\tanh(t/2)$,

$$\cosh\frac{t}{2}P^{\pm}(\xi,t)=\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n)!!}(-1)^n A_n^{\pm}(\xi)\left(\tanh\frac{t}{2}\right)^{2n+1\mp 1}, \tag{2.8}$$

where $A_n^{\pm}(\xi)$ are expressed via the Fourier integrals containing powers of $\tanh(t/2)$:

$$A_n^{\pm}(\xi)=\frac{\sinh\pi\xi}{4\pi\xi}\int_{-\infty}^{\infty}\frac{(i\tanh(x/2))^{2n+1/2\mp 1/2}}{\cosh^2(x/2)}\exp(-i\xi x)\,dx. \tag{2.9}$$

In order to find a generating function for $A_n^{\pm}(\xi)$, we use the identity

$$\exp(i\xi x)=\frac{1}{\cosh^2(x/2)}\sum_{n=0}^{\infty}(n+1)p_n(\xi)\left(i\tanh\frac{x}{2}\right)^n, \tag{2.10}$$

obtained from Eqs. (18)–(21) in [1, Vol. 2, Section 10.21], where

$$p_n(\xi)=\frac{P_n^1(\xi,\pi/2)}{n+1}=i^nF(-n,1+i\xi;2;2)$$

is a modified Pollaczek polynomial satisfying

$$\begin{aligned}&(n+1)p_n(\xi)-2\xi p_{n-1}(\xi)+(n-1)p_{n-2}(\xi)=0,\quad n=2,3,\ldots,\\&p_0(\xi)=1,\qquad p_1(\xi)=\xi,\end{aligned} \tag{2.11}$$

and being orthogonal:

$$\int_{-\infty}^{\infty}\frac{\xi}{\sinh\pi\xi}p_n(\xi)p_m(\xi)\,d\xi=\begin{cases}\frac{1}{2(n+1)} & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases} \tag{2.12}$$

Substituting this expression into the integral representations (2.3) and integrating term-by-term we ascertain that

$$\cosh\frac{t}{2}P^{\pm}(\xi,t)=\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n)!!}(-1)^n p_{2n+1/2\mp 1/2}(\xi)\left(\tanh\frac{t}{2}\right)^{2n+1\mp 1}. \tag{2.13}$$

Hence $A_n^+(\xi)=p_{2n}(\xi)$, $A_n^-(\xi)=p_{2n+1}(\xi)$.

Evaluating the Fourier integral (2.9) yields another expression involving the hypergeometric function

$$p_n(\xi)=\operatorname{Re}\frac{i^n n!}{\Gamma(1+i\xi)\Gamma(n+2-i\xi)}F(n+1,-i\xi;n+2-i\xi;-1). \tag{2.14}$$

If we integrate (2.10) with respect to x from 0 to x, then

$$\exp(i\xi x)=1+2\xi\sum_{n=1}^{\infty}p_{n-1}(\xi)\left(i\tanh\frac{x}{2}\right)^n. \tag{2.15}$$

This generating function is a source of a number of useful formulas. One of them is deduced from the identity $\exp(\xi_1+\xi_2)=\exp(\xi_1)\exp(\xi_2)$ by multiplying two power series of form (2.15):

$$(\xi_1+\xi_2)p_n(\xi_1+\xi_2)=2\xi_2\xi_1\sum_{k=1}^{n}p_{k-1}(\xi_1)p_{n-k}(\xi_2)+\xi_1 p_n(\xi_1)+\xi_2 p_n(\xi_2). \tag{2.16}$$

Let $\xi_1=(1/2)(\eta-\zeta)$, $\xi_2=(1/2)(\eta+\zeta)$. Then

$$\begin{aligned}2\eta p_n(\eta)&=(\eta^2-\zeta^2)\sum_{k=1}^{n}p_{k-1}\left(\frac{\eta-\zeta}{2}\right)p_{n-k}\left(\frac{\eta+\zeta}{2}\right)\\&\quad+(\eta-\zeta)p_n\left(\frac{\eta-\zeta}{2}\right)+(\eta+\zeta)p_n\left(\frac{\eta+\zeta}{2}\right).\end{aligned} \tag{2.17}$$

As $n=2m+1$, $\eta=0$, and $\zeta=2\xi$, we derive from this formula

$$p_{2m+1}(\xi)=(-1)^m\xi p_m^2(\xi)+2\xi\sum_{k=0}^{m-1}(-1)^k p_k(\xi)p_{2m-k}(\xi). \tag{2.18}$$

Using this relation, we obtain, as $\eta=2\xi$, $\zeta=0$,

$$\begin{aligned}p_{2m+1}(2\xi)&=2\xi\sum_{k=0}^{[m]/2}p_{2k}(\xi)p_{2m-2k}(\xi),\\p_{2m}(2\xi)&=p_{2m}(\xi)+2\xi\sum_{k=0}^{m-1}p_{2k}(\xi)p_{2m-2k-1}(\xi).\end{aligned} \tag{2.19}$$

Now let $a_{nk}(s)$, $s=2,3,\ldots$, be coefficients of the expansion

$$\left[\frac{(1+z)^s-(1-z)^s}{z[(1+z)^s+(1-z)^s]}\right]^n=\sum_{k=0}^{\infty}a_{nk}(s)z^{2k}.$$

Then

$$\tanh^n\frac{sx}{2}=\sum_{k=[n/2]}^{\infty}a_{n,k-[n/2]}(s)\left(\tanh\frac{x}{2}\right)^{2k+\lambda_n},\quad \lambda_n=\frac{1-(-1)^n}{2},$$

and after that

$$\begin{aligned}\exp(i\xi sx)-1&=2s\xi\sum_{k=1}^{\infty}p_{k-1}(s\xi)\left(i\tanh\frac{x}{2}\right)^k=2\xi\sum_{n=1}^{\infty}p_{n-1}(\xi)\left(i\tanh\frac{sx}{2}\right)^n\\&=2\xi\sum_{k=1}^{\infty}\left(\sum_{n=1}^{[(k+1)/2]}(-1)^{n+[(k+1)/2]}a_{2n-\lambda_k,[(k+1)/2]-n}(s)p_{2n-\lambda_k-1}(\xi)\right)\\&\quad\times\left(i\tanh^k\frac{x}{2}\right)^k.\end{aligned}$$

This implies

$$p_{k-1}(s\xi) = \frac{1}{s}\sum_{n=1}^{[(k+1)/2]} (-1)^{n+[(k+1)/2]} a_{2n-\lambda_k,[(k+1)/2]-n}(s)\, p_{2n-\lambda_k-1}(\xi),$$
$$s = 2, 3, \ldots. \tag{2.20}$$

In particular,

$$p_{k-1}(2\xi) = \sum_{n=1}^{[(k+1)/2]} \frac{2^{2n-\lambda_k-1}([k/2]+n-1)!}{(2n-\lambda_k-1)!([(k+1)/2]-n)!} p_{2n-\lambda_k-1}(\xi). \tag{2.21}$$

To determine an expression for derivatives of $p_n(\xi)$, we write the identity

$$\frac{\partial^l}{\partial \xi^l} \exp(i\xi x) = (ix)^l \exp(i\xi x),$$

replace $\exp(i\xi x)$ by (2.15), and make use of the series

$$(ix)^l = \left(2i \tanh\frac{x}{2}\sum_{k=0}^{\infty}\frac{\tanh^{2k}(x/2)}{2k+1}\right)^l = \sum_{m=l}^{\infty} b_{l,m-l}\left(i\tanh\frac{x}{2}\right)^m.$$

Here

$$b_{l,m} = \begin{cases} 0, & m = 2k+1, \\ 2^l(-1)^k c_{l,k}, & m = 2k, \end{cases}$$
$$c_{l,0} = 1, \qquad c_{l,k} = \frac{1}{k}\sum_{s=1}^{k}\frac{(sl-k+s)}{2s+1} c_{l,k-s}. \tag{2.22}$$

Equating coefficients of like powers of $i\tanh(x/2)$ yields a formula for derivatives of $p_n(\xi)$:

$$\frac{d^l}{d\xi^l}\big[\xi p_n(\xi)\big] = \xi\sum_{k=0}^{n-l} p_k(\xi) b_{l,n-l-k} + \frac{1}{2} b_{l,n-l+1}. \tag{2.23}$$

The both sides of (2.23) are identical if

$$p_n(\xi) = \frac{1}{2}\sum_{l=0}^{n}\frac{b_{l+1,n-l}}{(l+1)!}\xi^l. \tag{2.24}$$

We apply series (2.13) and (2.24) to evaluate the integrals

$$\mathcal{A}_m^{\pm} = \int_{-\infty}^{\infty}\frac{\xi^{m+1}}{\sinh\pi\xi} P_{i\xi}(\cosh t)\, d\xi = i^{1/2\mp 1/2}\int_0^{\infty}\frac{2\xi^{m+1}}{\sinh\pi\xi} P^{\pm}(\xi t)\, d\xi,$$
$$m = 2k + \frac{1}{2} \mp \frac{1}{2},\ k = 0, 1, \ldots. \tag{2.25}$$

Integrating term-by-term, one can obtain on account of orthogonality of the polynomials $p_n(\xi)$:

$$\mathcal{A}_m^{\pm} = i^{1/2\mp 1/2} \frac{\tanh(t/2)}{\cosh(t/2)} \sum_{n=0}^{[m/2]} (-1)^n g_{m,N}^{\pm} \frac{(2n+1)!!}{(2n)!!} \left(\tanh \frac{t}{2}\right)^{2n\mp 1},$$

$$g_{m,N}^{\pm} = 2 \int_0^{\infty} \frac{\xi^{m+1}}{\sinh \pi \xi} p_N(\xi)\, d\xi, \quad N = 2n + \frac{1}{2} \mp \frac{1}{2}. \tag{2.26}$$

To determine the coefficients $g_{m,N}^{\pm}$, we replace $p_N(\xi)$ by (2.24) and make use of the integral representation of the Bernoulli numbers [1, Vol. 1, p. 53, Eq. (26)]. Finally

$$g_{m,N}^{\pm} = \sum_{l=0}^{N} \frac{2^{l+m+2} - 1}{2(l+m+2)(l+1)!} |B_{l+m+2}| b_{l+1,N-l}, \tag{2.27}$$

where $b_{l,k}$ are defined above, B_k are the Bernoulli numbers.

3. Asymptotic expansions

As $\xi \gg 1$, asymptotic expansions of $P^{\pm}(\xi, t)$ can be derived from the integral representations (2.3). Let

$$R(t,x) = \cosh \frac{x}{2} \sqrt{\frac{t^2 - x^2}{\cosh t - \cosh x}} = \sum_{n=0}^{N} (-1)^n a_n(t)(t^2 - x^2)^n + R_1(t,x), \tag{3.1}$$

where $a_n(t)$ are the Tailor coefficients of the function $R(t, \sqrt{s})$ at the point $s = t^2$. It is readily seen that $R_0(x) = R_1(t,x)(t^2 - x^2)^{-1/2}$ is an even continuous function on $[-t, t]$. $R_0(x)$ possesses N derivatives $R_0^{(n)}(x)$ which are continuous on $[-t, t]$ and $R_0^{(n)}(\pm t) = 0$. $R_0^{(N+1)}(x)$ is integrable function. Integrating by parts, we arrive at the estimate

$$\int_0^t R_0(x) \cos \xi x\, dx = \frac{1}{2} \int_{-t}^{t} R_0(x) \exp(i\xi x)\, dx = o(\xi^{-N-1}).$$

Insert (3.1) into (2.3) and evaluate the resulting integrals. The integral representation of the Bessel function of the first kind $J_n(\xi t)$ [1, Vol. 2, Section 7.3.4, Eq. (11)] give us an uniform asymptotic expansion for $t \geqslant 0$, $\xi \gg 1$:

$$P^{+}(\xi, t) \sim \frac{1}{\sqrt{2}} \sum_{n=0}^{N} (-1)^n (2n-1)!! a_n(t) \frac{t^n}{\xi^n} J_n(\xi t) + o(\xi^{-N-1}),$$

$$a_0(t) = \sqrt{t \coth \frac{t}{2}}, \qquad a_1(t) = \frac{a_0(t)}{8t} \left(\frac{\cosh t - 2}{\sinh t} + \frac{1}{t} \right). \tag{3.2}$$

Asymptotic formulas for the Bessel functions [1, Vol. 2, Section 7.13.1, Eq. (3)] yield the more convenient asymptotic expansion that is valid as $\xi \gg 1$, $t \geqslant t_0 > 0$:

$$P^{+}(\xi,t)\sim\sqrt{\frac{\coth(t/2)}{\pi\xi}}\left[\cos\left(\xi t-\frac{\pi}{4}\right)+\frac{(2-\cosh t)\sin(\xi t-\pi/4)}{8\xi\sinh t}\right]$$
$$+O\left(\frac{1}{\xi^{5/2}}\right). \tag{3.3}$$

Asymptotic expansions for $P^{-}(\xi,t)$ can be obtained in the same way. As $t\geqslant 0$, $\xi\gg 1$,

$$P^{-}(\xi,t)\sim\frac{1}{\sqrt{2}}\sum_{n=0}^{N}(-1)^{n}(2n-1)!!b_{n}(t)\frac{t^{n+1}}{\xi^{n}}J_{n+1}(\xi t)+o(\xi^{-N-1}),$$
$$b_0(t)=\sqrt{t^{-1}\tanh\frac{t}{2}},\qquad b_1(t)=-\frac{b_0(t)}{8t}\left(\frac{3}{t}-\frac{2+\cosh t}{\sinh t}\right), \tag{3.4}$$

where $b_n(t)$ are the Taylor coefficients of the function

$$(t^2-s)^{1/2}(\cosh t-\cosh\sqrt{s}\,)^{-1/2}s^{-1/2}\sinh\frac{\sqrt{s}}{2}$$

at the point $s=t^2$. This implies

$$P^{-}(\xi,t)\sim\sqrt{\frac{\tanh(t/2)}{\pi\xi}}\left[\sin\left(\xi t-\frac{\pi}{4}\right)+\frac{(2+\cosh t)\cos(\xi t-\pi/4)}{8\xi\sinh t}\right]$$
$$+O\left(\frac{1}{\xi^{5/2}}\right). \tag{3.5}$$

The asymptotic formula obtained is valid for $t\geqslant t_0>0$, $\xi\gg 1$.

4. Integral transforms

Definition 4.1. The set of all even (odd) functions $f(t)\in C^{\infty}(0,\infty)$ satisfying for any non-negative integers r and m the condition $\sup|t^m f^{(r)}(t)|<\infty$ with the norm $\|f(t)\|=\int_0^\infty tf^2(t)\coth\pi t\,dt$ is denoted $\mathfrak{S}^{+}$ ($\mathfrak{S}^{-}$).

Definition 4.2. The set of all even functions $f(t)\in C^{\infty}(0,\infty)$ satisfying for any non-negative integers r and m the condition $\sup|t^m f^{(r)}(t)|<\infty$ with the norm $\|f(t)\|^{+}=\int_0^\infty f^2(t)\tanh(t/2)\,dt$ is denoted $\mathfrak{G}^{+}$. If $f(t)\in\mathfrak{G}^{+}$, then the set of functions of the form $g(t)=\tanh(t/2)f'(t)$ with the norm $\|g(t)\|^{-}=\int_0^\infty g^2(t)\coth(t/2)\,dt$ is denoted $\mathfrak{G}^{-}$.

Let $f(t)\in\mathfrak{G}^{+}$ and $\varphi(t)$ be the function defined by the expression

$$\varphi(t)=-\frac{1}{\pi\cosh(t/2)}\frac{d}{dt}\int_t^\infty\frac{\sinh x}{\sqrt{\cosh x-\cosh t}}f(x)\,dx$$
$$=-\frac{2}{\pi}\sinh\frac{t}{2}\int_t^\infty\frac{f'(x)\,dx}{\sqrt{\cosh x-\cosh t}},\quad t\geqslant 0. \tag{4.1}$$

The function $\varphi(t)$ is the unique solution of the Abel equation [7]

$$\int_t^\infty \frac{\cosh(x/2)}{\sqrt{\cosh x - \cosh t}}\varphi(x)\,dx = f(t). \tag{4.2}$$

Upon integrating by parts and differentiating one might establish that $\varphi(t) \in \mathfrak{S}^-$. Thus formulas (4.1) and (4.2) establish one-to-one correspondence for functions belonging to $\mathfrak{G}^+$ and $\mathfrak{S}^-$.

Theorem 4.3. *The linear integral transforms*

$$F^+(\xi) = \sqrt{2}\int_0^\infty f(x)P^+(\xi,x)\tanh\frac{x}{2}\,dx, \quad f(t) \in \mathfrak{G}^+, \tag{4.3}$$

$$f(t) = \sqrt{2}\int_0^\infty F^+(\xi)P^+(\xi,t)\xi\coth\pi\xi\,d\xi, \quad F^+(\xi) \in \mathfrak{S}^+, \tag{4.4}$$

are reciprocal continuous and isometric mappings

$$\|f(t)\|^+ = \|F^+(t)\|. \tag{4.5}$$

Proof. Represent the function $\varphi\ (x) \in \mathfrak{S}^-$ introduced above in the form of the Fourier sine integral:

$$\varphi(x) = \frac{2}{\pi}\int_0^\infty \hat{\varphi}(\xi)\sin\xi x\,d\xi, \tag{4.6}$$

$$\hat{\varphi}(\xi) = \int_0^\infty \varphi(t)\sin\xi t\,dt. \tag{4.7}$$

It is well known that in this case $\hat{\varphi}(\xi) \in \mathfrak{S}^-$. Multiply (4.6) by

$$\left[\cosh\frac{x}{2}(\cosh x - \cosh t)^{-1/2} - 2^{-1/2}\right]$$

and integrate with respect to x from t to ∞. Then the Fubini theorem enables us to interchange the order of integration. We obtain on the basis of (4.2) and (2.4) that

$$f(t) - \frac{1}{\sqrt{2}}\int_t^\infty \varphi(x)\,dx = \sqrt{2}\int_0^\infty \hat{\varphi}(\xi)P^+(\xi,x)\coth\pi\xi\,d\xi - \frac{\sqrt{2}}{\pi}\int_0^\infty \hat{\varphi}(\xi)\frac{\cos\xi t}{\xi}\,d\xi$$

or

$$f(t) = \sqrt{2}\int_0^\infty \hat{\varphi}(\xi)P^+(\xi,x)\coth\pi\xi\,d\xi, \quad f(t) \in \mathfrak{G}^+. \tag{4.8}$$

Now we shall express $\hat{\varphi}(\xi)$ in terms of $f(t)$. This can be carried out by inserting (4.2) into (4.7). Because $\varphi(x) \in \mathfrak{S}^-$, we are permitted to interchange the order of integration and find

$$\hat{\varphi}(\xi) = -\frac{2}{\pi} \int_0^\infty \left(\int_0^x \frac{\sinh(t/2) \sin \xi t}{\sqrt{\cosh x - \cosh t}} \, dt \right) df(x) = -\sqrt{2} \int_0^\infty P^-(\xi, x) \, df(x). \tag{4.9}$$

Finally, integration by parts yields

$$\hat{\varphi}(\xi) = \sqrt{2}\xi \int_0^\infty f(x) P^+(\xi, x) \tanh \frac{x}{2} \, dx = \xi F^+(\xi). \tag{4.10}$$

It is readily seen that (4.10) and (4.8) are equivalent to (4.3) and (4.4).

Isometry (the Plancherel formula) is proved upon putting (4.4) into left-hand side of (4.5) and interchanging the order of integration. □

Theorem 4.4. *The linear integral transforms*

$$G^-(\xi) = \sqrt{2} \int_0^\infty g(x) P^-(\xi, x) \coth \frac{x}{2} \, dx, \quad g(x) \in \mathfrak{G}^-, \tag{4.11}$$

$$g(t) = \sqrt{2} \int_0^\infty G^-(\xi) P^-(\xi, t) \xi \coth \pi \xi \, d\xi, \quad G^-(\xi) \in \mathfrak{S}^-, \tag{4.12}$$

are reciprocal continuous and isometric mappings

$$\|g(x)\|^- = \|G^-(t)\|. \tag{4.13}$$

Proof. Substituting $g(t) = -f'(t) \tanh(t/2)$, $\hat{\varphi}(\xi) = G^-(\xi)$ into (4.9) leads to (4.11). Then the relation (4.12) is obtained by differentiating the formula (4.8). Isometry may be proved as in the preceding theorem. □

The theory of the integral transforms introduced in L_2 is analogous to the Plancherel theory of Fourier transforms [8,9].

Theorem 4.5. *Let* $f(t)(\tanh(t/2))^{\pm 1/2} \in L_2(0, \infty)$. *Then integral transforms*

$$F^\pm(\xi) = \underset{A \to \infty}{\text{l.i.m.}} \sqrt{2} \int_0^A f(x) P^\pm(\xi, x) \left(\tanh \frac{x}{2} \right)^{\pm 1} dx, \tag{4.14}$$

$$f(t) = \underset{A \to \infty}{\text{l.i.m.}} \sqrt{2} \int_0^A F^\pm(\xi) P^\pm(\xi, t) \xi \coth \pi \xi \, d\xi \tag{4.15}$$

satisfy

$$\int_0^\infty f^2(x)\left(\tanh\frac{x}{2}\right)^{\pm 1} dx = \int_0^\infty \left[F^{\pm}(\xi)\right]^2 \xi \coth \pi\xi \, d\xi. \tag{4.16}$$

The formulas

$$f(t) = \sqrt{2}\coth\frac{t}{2}\frac{d}{dt}\int_0^\infty F^{+}(\xi)P^{-}(\xi,t)\coth\pi\xi\,d\xi, \tag{4.17}$$

$$f(t) = \sqrt{2}\tanh\frac{t}{2}\frac{d}{dt}\int_0^\infty F^{-}(\xi)\left[P^{+}(\xi,t_0) - P^{+}(\xi,t)\right]\coth\pi\xi\,d\xi \tag{4.18}$$

are valid at every point where $f(t)$ is a derivative of its integral.

Proof. Because $\mathfrak{S}^{\pm}$ and $\mathfrak{G}^{\pm}$ are dense sets in $L_2(0,\infty)$, relations (4.14)–(4.16) can be proved, by virtue of the previous theorems, in a manner of the Plancherel theory. To deduce formulas (4.17) and (4.18), we note that the formula

$$\int_0^\infty f(x)g(x)\left(\tanh\frac{x}{2}\right)^{\pm 1} dx = \int_0^\infty F^{\pm}(\xi)G^{\pm}(\xi)\xi\coth\pi\xi\,d\xi \tag{4.19}$$

follows from (4.16) as $g(t)(\tanh(t/2))^{\pm 1/2} \in L_2(0,\infty)$. Let $t_0 > 0$,

$$g(x) = \begin{cases} 1 & \text{if } x \in [t_0, t], \\ 0 & \text{if } x \notin [t_0, t]. \end{cases} \tag{4.20}$$

The integral transforms of this function can be readily evaluated by means of (2.1)

$$G^{+}(\xi) = \sqrt{2}\frac{P^{-}(\xi,t) - P^{-}(\xi,t_0)}{\xi}, \qquad G^{-}(\xi) = \sqrt{2}\frac{P^{+}(\xi,t_0) - P^{+}(\xi,t)}{\xi}. \tag{4.21}$$

Now (4.22) becomes

$$\int_{t_0}^{t} f(x)\left(\tanh\frac{x}{2}\right)^{\pm 1} dx = \int_0^\infty F^{\pm}(\xi)G^{\pm}(\xi)\xi\coth\pi\xi\,d\xi. \tag{4.22}$$

This implies (4.17), (4.18) since we can set $t_0 = 0$ for $G^{+}(\xi)$. $\square$

Remark 1. We cannot set $t_0 = 0$ for $G^{-}(\xi)$ without any additional justification for fear that $g(t)(\tanh(t/2))^{-1/2} \notin L_2(0,\infty)$.

Theorem 4.6. *If $f(t)\tanh t \in L_1(0,\infty)$ $(f(t) \in L_1(0,\infty))$, then formula* (4.17) ((4.18)) *is valid at every point $t \in (0,\infty)$, where $f(t)$ is the derivative of its integral.*

Proof. Using the Fubini theorem we write

$$\int_0^\infty F^+(\xi)P^-(\xi,t)\coth\pi\xi\,d\xi$$

$$= \lim_{A\to\infty}\int_0^\infty f(x)\tanh\frac{x}{2}\left(\int_0^A P^+(\xi,x)P^-(\xi,t)\coth\pi\xi\,d\xi\right)dx. \tag{4.23}$$

By direct calculation based on the asymptotic expansions (3.2) and (3.4), one might ascertain that the inner integral has finite bounds for $A \geqslant A_0 > 0$, $x \geqslant 0$, and any fixed $t > 0$. This permits us to look for the limit $A \to \infty$ inside the integral. Then (4.22), (4.21) and (4.20) yield

$$2\int_0^\infty P^+(\xi,x)P^-(\xi,t)\coth\pi\xi\,d\xi = 2\coth\frac{x}{2}\frac{d}{dx}\int_0^\infty \frac{P^-(\xi,t)}{\xi}\frac{P^-(\xi,x)}{\xi}\xi\coth\pi\xi\,d\xi$$

$$= \coth\frac{x}{2}\frac{d}{dx}\int_0^{\min(x,t)}\tanh\frac{s}{2}\,ds = \begin{cases} 0, & x > t, \\ 1, & x < t. \end{cases} \tag{4.24}$$

Utilizing this formula and differentiating (4.23), we arrive at (4.17).

The formula (4.18) can be proved in a completely analogous manner. □

Theorem 4.7. *Let $f(t) \in L_1$ over any interval (ε,∞), $\varepsilon > 0$, and $f_\varepsilon(t)$ be an integral mean of $f(t)$ defined by the relation*

$$f\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(s)\,ds, & t > \varepsilon, \\ 0, & 0 < t \leqslant \varepsilon. \end{cases}$$

Then the formula

$$f(t) = \sqrt{2}\lim_{\varepsilon\to 0}\int_0^\infty F_\varepsilon^\pm(\xi)P^\pm(\xi,t)\xi\coth\pi\xi\,d\xi \tag{4.25}$$

is valid at every point $t \in (0,\infty)$ where $f(t)$ is the derivative of its integral.

Proof. Upon integrating by parts we obtain

$$\frac{\xi}{\sqrt{2}}F_\varepsilon^+(\xi) = \frac{\xi}{\varepsilon}\int_\varepsilon^\infty P^+(\xi,x)\tanh\frac{x}{2}\left(\int_x^{x+\varepsilon} f(s)\,ds\right)dx$$

$$= \int_{2\varepsilon}^\infty f(x)\big[P^-(\xi,\varepsilon) - P^-(\xi,x-\varepsilon)\big]\,dx$$

$$+ \int_\varepsilon^\infty f(x)\big[P^-(\xi,x) - P^-(\xi,\varepsilon)\big]\,dx.$$

Insert this into the right-hand part of (4.25), where $t > \varepsilon$, and interchange the order of integration. This operation can be proved to be correct similarly to the previous theorem. Then (4.24) leads to

$$\sqrt{2}\lim_{\varepsilon\to 0}\int_0^\infty F_\varepsilon^+(\xi)P^+(\xi,t)\xi\coth\pi\xi\,d\xi = \lim_{\varepsilon\to 0} f_\varepsilon(t) = f(t).$$

The case of the transform with the kernel $P^-(\xi,t)$ may be treated in the same way. □

Theorem 4.8. *Let $f(t)(\sqrt{\tanh t}\,)^{\pm 1} \in L_1(0,\infty)$. If the function $f(t)$ is of bounded variation in the vicinity of the point $t = x$, $x > 0$, then the formula*

$$\frac{f(x+0)+f(x-0)}{2} = \sqrt{2}\int_0^\infty F^\pm(\xi)P^\pm(\xi,x)\xi\coth\pi\xi\,d\xi \tag{4.26}$$

is valid.

Proof. We commence with the case of a function being continuous at $t = x$. In this case the formula (4.17) is valid. We write

$$\begin{aligned}\sqrt{2}&\int_0^\infty F^+(\xi)P^+(\xi,x)\xi\coth\pi\xi\,d\xi\\ &= \int_0^\infty f(s)\tanh\frac{s}{2}\left[\int_0^\infty Q(\xi,x,s)\xi\,d\xi\right]ds\\ &\quad + \sqrt{x\coth\frac{x}{2}}\int_0^\infty \xi J_0(\xi x)\left(\int_0^\infty \sqrt{s\tanh\frac{s}{2}}\,f(s)J_0(\xi s)\,ds\right)d\xi,\end{aligned} \tag{4.27}$$

where the left-hand part is the result of differentiating formally in (4.17) and

$$Q(\xi,x,s) = 2P^+(\xi,x)P^+(\xi,s)\xi\coth\pi\xi - \sqrt{xs\coth\frac{x}{2}\coth\frac{s}{2}}\,J_0(\xi s)J_0(\xi s).$$

In the right-hand part of (4.27), the second integral converges uniformly with respect to x by virtue of the well-known theorem for the Hankel integral transform ([10, Section XIV], [1, Vol. 2, Section 7.10.5]). The first integral is uniformly convergent as well. This may be proved by invoking the asymptotic formula (3.2). Thus one is really permitted to differentiate the integral in (4.17) inside the integration sign and to obtain the desired result.

As the function $f(t)$ is not continuous at $t = x$, it is possible to write $f(t) = f_1(t) + g(t)$, where $g(t)$ is a continuous function at the point $t = x$:

$$f_1(t) = f(x+0)\big[H(t-x) - H(t-x-1)\big] + f(x-0)H(x-t),$$

where $H(t)$ is the Heaviside unit step function. As it was established above, the theorem holds for $g(t)$. Then the theorem must be proved for $f_1(t)$. We get

$$\frac{1}{\sqrt{2}}F_1^+(\xi) = \left[f(x-0) - f(x+0)\right]P^-(\xi,x) + f(x+0)P^-(\xi,x+1).$$

Now a direct calculation yields

$$\sqrt{2}\int_0^\infty F_1^+(\xi)P^+(\xi,x)\xi\coth\pi\xi\,d\xi = \frac{f(x+0)+f(x-0)}{2}. \tag{4.28}$$

Formula (4.24) and the integral

$$2\int_0^\infty P^+(\xi,x)P^-(\xi,x)\coth\pi\xi\,d\xi = \coth\frac{x}{2}\frac{d}{dx}\int_0^\infty\left(\frac{P^-(\xi,x)}{\xi}\right)^2\xi\coth\pi\xi\,d\xi$$

$$= \frac{1}{2}\coth\frac{x}{2}\frac{d}{dx}\int_0^x\tanh\frac{s}{2}\,ds = \frac{1}{2}, \quad x>0, \tag{4.29}$$

have been exploited to obtain (4.28). Since $g(x)=0$, this give us (4.26).

For the integral transforms with the kernel $P^-(\xi,s)$ the theorem may be proved in a similar fashion. □

We note in conclusion that in many cases for approximative evaluation of the inverse transforms it is helpful to have recourse to integral (2.26).

Integral transforms involving the Legendre function of imaginary degree can be established by combining the integral transforms introduced whose inverses are rewritten in the form

$$f(t) = \frac{1}{\sqrt{2}i^{1/2\mp1/2}}\int_{-\infty}^{\infty}F^\pm(\xi)P_{-i\xi}(\cosh t)\xi\coth\pi\xi\,d\xi.$$

Let now $f(t)=g(t)$, $\xi[F^+(\xi)+iG^-(\xi)] = -i\sqrt{2}\Omega(\xi)$. Since $f(t)=(1/2)[f(t)+g(t)]$, we have the following theorem:

Theorem 4.9. *Let $f(t)\sqrt{\coth(t/2)}\in L_1(0,\infty)$. If the function $f(t)$ is of bounded variation in the vicinity of the point $t=x$, $x>0$, then the formulas*

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2i}\int_{-\infty}^{\infty}\Omega(\xi)P_{-i\xi}(\cosh x)\coth\pi\xi\,d\xi, \tag{4.30}$$

$$\Omega(\xi) = \int_0^\infty f(t)P_{i\xi}^1(\cosh t)\,dt \tag{4.31}$$

are valid.

In the foregoing, we have used the relation

$$P^1_{i\xi}(\cosh t) = \frac{d P_{i\xi}(\cosh t)}{dt} = i\xi\left[\tanh\frac{t}{2}P^+(\xi,t) + i\coth\frac{t}{2}P^-(\xi,t)\right],$$

where $P^1_{i\xi}(s)$ is the associated Legendre function.

Another form of the inversion formula is obtained by utilizing the identity $f(t) = g(t)\cosh^2(t/2) - f(t)\sinh^2(t/2)$:

$$f(x) = \frac{\sinh x}{2}\int_{-\infty}^{\infty}\frac{1}{\xi}\Omega(\xi)P^1_{i\xi}(\cosh x)\coth\pi\xi\,d\xi. \tag{4.32}$$

Formally, ignoring restrictions placed on the parameters, this is the special case of the generalized Mehler–Fok transform which was considered by Vilenkin [4,5] for functions belonging to $L_2(0,\infty)$.

When $f(t) = \psi(t)\cosh^2(t/2)$, $g(t) = \psi(t)\sinh^2(t/2)$, $F^+(\xi) + iG^-(\xi) = \sqrt{2}\Psi(\xi)$, we arrive at the integral transform

$$\Psi(\xi) = \int_0^{\infty}\psi(t)\sinh t\,P_{i\xi}(\cosh t)\,dt, \tag{4.33}$$

$$\psi(x) = \frac{1}{2}\int_{-\infty}^{\infty}\xi\Psi(\xi)P_{i\xi}(\cosh x)\coth\pi\xi\,d\xi. \tag{4.34}$$

This is the special case of the generalized Mehler–Fok transform as well.

The classical Mehler–Fok transform require the very restricted class of functions $t\exp(t/2)f(t) \in L_1(0,\infty)$ [6]. Mehler–Fok transform for functions from $L_2(0,\infty)$ cannot be applied to boundary value problems with conditions at edges whose solutions do not belong to $L_2(0,\infty)$. The essential advantage of the integral transforms (4.30)–(4.31) is a less restrictive class of functions. For this reason, these transforms are convenient for studying various boundary value problems usually studied with Mehler–Fok transforms.

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